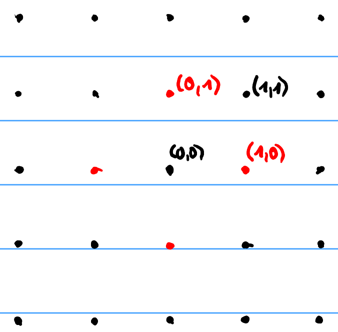
1. Affine varieties

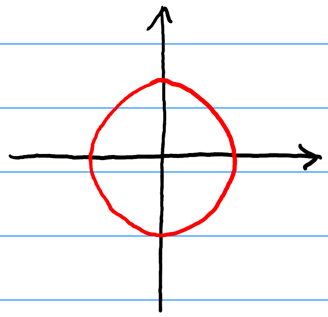
<u>Slogan</u> Algebraic Geometry studies solution sets of systems of polynomial equations in finitely many variables.

<u>Example</u> Equations $= \{x^2+y^2=1\}$

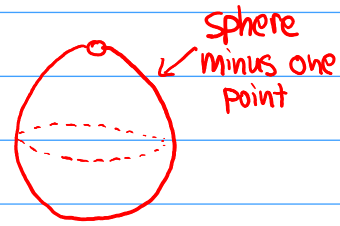
Solutions depend on allowed values for x,y!



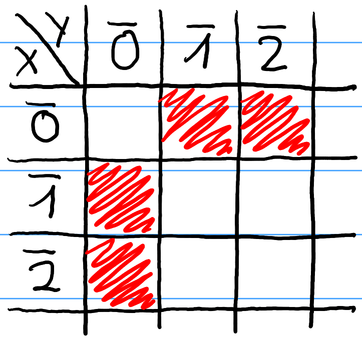
$(x,y) \in \mathbb{Z}^2$



$(x,y) \in \mathbb{R}^2$



$(x,y) \in \mathbb{C}^2$



$(x,y) \in \mathbb{F}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

<u>For now</u> Solutions over algebraically closed field $K = \overline{K}$.

<u>Def</u> (Affine varieties)

(a) We call
$$\mathbb{A}^n := \mathbb{A}^n_K = \{(a_1, \dots, a_n) : a_i \in K \text{ for } i = 1, \dots, n\}$$
the <u>affine n-space</u> over K.

(b) For a subset $S \subseteq K[x_1, \dots, x_n]$ of polynomials we call
$$V(S) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\}$$
the (affine) <u>zero locus</u> of S. Subsets of $\mathbb{A}^n$ of this form are called <u>affine varieties</u>.

<u>Notation</u>

$$V(f_1, \dots, f_m) := V(\{f_1, \dots, f_m\})$$

Examples of affine varieties

(a) $\mathbb{A}^n = V(0)$, $\emptyset = V(1)$

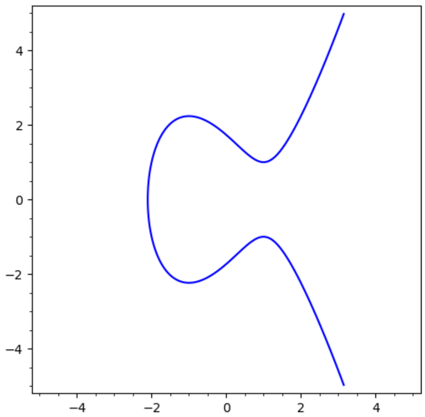
(b) $a = (a_1, \dots, a_n) \in \mathbb{A}^n$: $\{a\} = V(x_1 - a_1, \dots, x_n - a_n)$

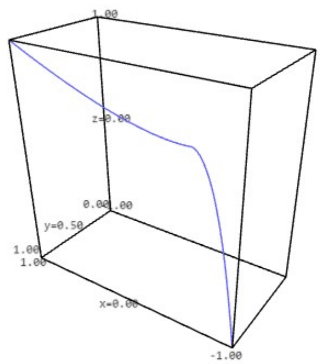
(c) Linear subspaces of $\mathbb{A}^n = K^n$

(d) $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ affine var.
$\Rightarrow X \times Y \subseteq \mathbb{A}^{n+m}$ affine variety $\Big\}$ Exercise

Fancy examples

Below, for $K = \mathbb{C}$ we draw the real solutions $V(S) \cap \mathbb{R}^n$.





$V(y^2 - x^3 + 3x - 3)$ $\qquad\qquad$ $V(x^3 - yz,\ y^2 - xz,\ z^2 - x^2 y)$

Lem (Properties of $V(\cdot)$)

(a) For any $S_1 \subseteq S_2 \subseteq K[x_1, \dots, x_n]$ we have $V(S_1) \supseteq V(S_2)$.

(b) For any $S_1, S_2 \subseteq K[x_1, \dots, x_n]$ we have $V(S_1) \cup V(S_2) = V(\underbrace{S_1 \cdot S_2}_{= \{fg \,:\, f \in S_1,\, g \in S_2\}})$.

(c) J index set, $S_i \subseteq K[x_1, \dots, x_n]$ for $i \in J$, then
$$\bigcap_{i \in J} V(S_i) = V\Big(\bigcup_{i \in J} S_i\Big).$$

In particular: finite unions & arbitrary intersections of affine varieties are again affine varieties.

Exercise Find an infinite union of affine varieties which is not an affine variety.

Proof (a) $x \in V(S_2) \Rightarrow \forall f \in S_2 : f(x) = 0$

$\xRightarrow{S_1 \subseteq S_2} \forall f \in S_1 : f(x) = 0 \Rightarrow x \in V(S_1)$.

(b) For all $x \in \mathbb{A}^n$:

$x \in V(S_1 \cdot S_2) \Leftrightarrow \forall f_1 \in S_1, f_2 \in S_2 : f_1(x) \cdot f_2(x) = 0 \in K$

$\underset{K \text{ domain}}{\Leftrightarrow} \forall f_1 \in S_1, f_2 \in S_2 : f_1(x) = 0 \text{ or } f_2(x) = 0$

$\Leftrightarrow (\forall f_1 \in S_1 : f_1(x) = 0) \text{ or } (\forall f_2 \in S_2 : f_2(x) = 0)$

$\Leftrightarrow x \in V(S_1) \cup V(S_2)$.

(c) $x \in \bigcap_{i \in J} V(S_i) \Leftrightarrow \forall i \in J : f(x) = 0 \quad \forall f \in S_i$

$\Leftrightarrow \forall f \in \bigcup_{i \in J} S_i : f(x) = 0 \Leftrightarrow x \in V(\bigcup_{i \in J} S_i)$ □

Exercise $\{\text{Affine varieties in } \mathbb{A}^1\} = \{\text{finite subsets of } \mathbb{A}^1\} \cup \{\mathbb{A}^1\}$

Rmk $S \subseteq K[x_1, \dots, x_n] \rightsquigarrow \langle S \rangle = \{ r_1 \cdot f_1 + \dots + r_m \cdot f_m : \begin{smallmatrix} r_i \in K[x_1, \dots, x_n] \\ f_i \in S \end{smallmatrix} \}$

ideal generated by S

Then:

$$V(S) = V(\langle S \rangle)$$

Pf "$\supseteq$" from $S \subseteq \langle S \rangle$ + Lemma

"$\subseteq$" $x \in V(S)$, $g = \sum r_i \cdot f_i \in \langle S \rangle$, with $f_i \in S$ $\Rightarrow f_i(x) = 0 \ \forall i \Rightarrow g(x) = 0$ $\Big\} \Rightarrow x \in V(\langle S \rangle)$. ※

Cor Any affine variety $X \subseteq \mathbb{A}^n$ can be written as

(a) $X = V(J)$, $\quad J \subseteq K[x_1, \dots, x_n]$ ideal

(b) $X = V(f_1, \dots, f_m)$, $\quad f_i \in K[x_1, \dots, x_n]$ finitely many polynomials.

Pf

$$X \overset{\text{Def}}{=\!=} V(S) \overset{\text{Rmk}}{=\!=} V(\underbrace{\langle S \rangle}_{=:J}) \xlongequal[\text{Basis Thm}]{\text{Hilbert's}} V(f_1, \dots, f_n)$$

$\hookrightarrow J = \langle f_1, \dots, f_n \rangle$ finitely generated. □

Lem (Properties of $V(\cdot)$ – ideal version)

For any ideals J, J_1, J_2 in $K[x_1, \dots, x_n]$ we have

(a) $V(\sqrt{J}) = V(J)$ $\quad \rightsquigarrow \sqrt{J} = \{ g \in K[x_1, \dots, x_n] : g^m \in J \text{ for some } m \}$ radical ideal

(b) $V(J_1) \cup V(J_2) = V(J_1 \cdot J_2) = V(J_1 \cap J_2)$

(c) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Pf (a) "$\subseteq$" since $J \subseteq \sqrt{J}$

"$\supseteq$" $x \in V(J)$, $g \in \sqrt{J} \Rightarrow \exists m \geq 0 : g^m \in J$

$\Rightarrow g(x)^m = 0$

$\Rightarrow g(x) = 0 \qquad \Rightarrow x \in V(\sqrt{J})$.

(b) First "=": Lemma

Second "=": from (a) since $\sqrt{J_1 \cdot J_2} = \sqrt{J_1 \cap J_2}$.

(c) $J_1 + J_2 \overset{(*)}{=} \langle J_1 \cup J_2 \rangle \Rightarrow V(J_1) \cap V(J_2) \overset{\text{Lem}}{=} V(J_1 \cup J_2)$

$\overset{\text{Rmk}}{=} V(\langle J_1 \cup J_2 \rangle)$

$\overset{(*)}{=} V(J_1 + J_2)$ $\square$

<u>Goal</u> Understand better the correspondence

$$\left\{ \begin{matrix} \text{Affine varieties} \\ X \subseteq \mathbb{A}^n \end{matrix} \right\} \underset{V(\cdot)}{\overset{??}{\rightleftarrows}} \left\{ \begin{matrix} \text{Ideals} \\ J \subseteq K[x_1, \dots, x_n] \end{matrix} \right\}$$

<u>Def</u> (Ideal of a subset of $\mathbb{A}^n$)

Let $X \subseteq \mathbb{A}^n$ be any subset. Then

$$I(X) = \{ f \in K[x_1, \dots, x_n] : f(x) = 0 \ \forall x \in X \}$$

is called the (vanishing) <u>ideal</u> of X.

$I(X)$ is an ideal: $f, f_1, f_2 \in I(X)$, $g \in K[x_1, \dots, x_n]$: $f \cdot g, f_1 + f_2 \in I(X)$.

<u>Rmk</u>

(a) $X_1 \subseteq X_2 \subseteq \mathbb{A}^n \Rightarrow I(X_1) \supseteq I(X_2)$

(b) $I(X) = \sqrt{I(X)}$ is radical: $f \in \sqrt{I(X)} \Rightarrow \exists m \geq 0 : f^m \in I(X)$

$\Rightarrow f(x)^m = 0 \ \forall x \in X$

$\Rightarrow f(x) = 0 \ \forall x \in X \Rightarrow f \in I(X)$.

Thus we have:

$$\left\{ \begin{matrix} \text{Affine varieties} \\ X \subseteq \mathbb{A}^n \end{matrix} \right\} \underset{V(\cdot)}{\overset{I(\cdot)}{\rightleftarrows}} \left\{ \begin{matrix} \text{Ideals} \\ J \subseteq K[x_1, \dots, x_n] \end{matrix} \right\}$$

<u>Q</u> What are the two compositions $I \circ V$ and $V \circ I$?

Nullstellensatz & consequences

Pro (Hilbert's Nullstellensatz)

(a) For any affine variety $X \subseteq \mathbb{A}^n$ we have $V(I(X)) = X$.

(b) For any ideal $J \subseteq K[x_1,\dots,x_n]$ we have $I(V(J)) = \sqrt{J}$.

In particular, we have

$$\left\{ \begin{matrix} \text{Ideals} \\ J \subseteq K[x_1,\dots,x_n] \end{matrix} \right\}$$

$$\cup\!\!|$$

$$\left\{ \begin{matrix} \text{Affine varieties} \\ X \subseteq \mathbb{A}^n \end{matrix} \right\} \underset{V(\cdot)}{\overset{I(\cdot)}{\rightleftarrows}} \left\{ \begin{matrix} \text{Radical ideals} \\ J \subseteq K[x_1,\dots,x_n] \end{matrix} \right\}$$

$\sim$ inclusion reversing bijection

Pf (a) "$\supseteq$": $x \in X \Rightarrow f(x) = 0 \ \forall f \in I(X) \Rightarrow x \in V(I(X))$

(**) (b) "$\supseteq$" $f \in \sqrt{J}$, $x \in V(J) \overset{\text{Lem}}{=} V(\sqrt{J}) \Rightarrow f(x) = 0 \Rightarrow f \in I(V(J))$.

(a) "$\subseteq$" X affine var. $\overset{\text{Def}}{\Longrightarrow} X = V(J)$

$$\Rightarrow V(I(X)) = V(\underbrace{I(V(J))}_{\overset{(**)}{\supseteq} \sqrt{J}}) \subseteq V(\sqrt{J}) = V(J) = X.$$

Difficult part (b) "$\subseteq$": If f vanishes on $V(J)$, then $\exists m$: $f^m \in J$.

$\rightsquigarrow$ this is Hilbert's Nullstellensatz from commutative algebra. $\square$

Example ($n=1$)

$J \subseteq K[x_1] \xrightarrow[\text{PID}]{K[x_1] \text{ is}} J = \langle f \rangle$ for $f = c \cdot (x_1 - a_1)^{k_1} \cdot (x_1 - a_2)^{k_2} \cdots (x_1 - a_r)^{k_r}$ ← K alg. closed

$\Rightarrow V(f) = \{a_1, \dots, a_r\} \subseteq \mathbb{A}^1$

$\Rightarrow I(V(f)) = \sqrt{J} = \langle (x_1 - a_1)(x_2 - a_2) \cdots (x_1 - a_r) \rangle$

Rmk K algebraically closed is necessary!

Eg. for $K = \mathbb{R}$, $J = (x^2+1) \subseteq \mathbb{R}[x] \Rightarrow V(J) = \emptyset$

$$\Rightarrow I(V(J)) = \mathbb{R}[x] \neq \sqrt{(x^2+1)} = (x^2+1)$$

prime ideal

Cor For K algebraically closed, we have a bijection:

$$\left\{\begin{array}{c}\text{points } a=(a_1,\dots,a_n)\\ \text{in } \mathbb{A}^n\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{maximal ideals}\\ m\subseteq K[x_1,\dots,x_n]\end{array}\right\}$$

$$a \longmapsto m_a=(x_1-a_1,\dots,x_n-a_n)=I(\{a\})$$

Pf Well-defined: $K[x_1,\dots,x_n]/m_a \xrightarrow{\sim} K$ is field $\Rightarrow m_a$ maximal

$$[f] \longmapsto f(a)$$

Injectivity: $V(m_a)=\{a\} \rightsquigarrow$ can reconstruct a from m_a

Surjectivity: m max. ideal in $K[x_1,\dots,x_n]$

Claim: $V(m)\neq\emptyset$. Otherwise: $I(V(m))=I(\emptyset)=K[x_1,\dots,x_n]$

$$\sqrt{m} \underset{\uparrow}{=} m \quad \neq \quad ⚡$$

m prime $\Rightarrow$ radical

Let $a\in V(m)$

$\{a\}\subseteq V(m) \Rightarrow I(\{a\})=m_a \supseteq m \xRightarrow{m \text{ maximal}} m=m_a$ $\square$

<u>Alternative proof</u>

For inclusion reversing bijection $\{\text{affine varieties}\} \rightleftarrows \{\text{radical ideals}\}$

$\rightarrow$ sets $\{a\}$ are minimal, non-empty affine varieties

$\rightarrow$ ideals $I(\{a\})$ are maximal, proper (radical) ideals $\square$

<u>Lem</u> (Properties of $I(\cdot)$) For any affine varieties X_1, X_2 in $\mathbb{A}^n$, we have

(a) $I(X_1\cup X_2)=I(X_1)\cap I(X_2)$

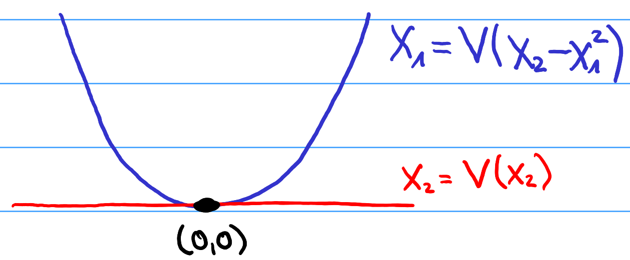
(b) $I(X_1\cap X_2)=\sqrt{I(X_1)+I(X_2)}$

Pf Exercise. ⁂

<u>Fun exercise</u>

Find examples of <u>subsets</u> $X_1, X_2\subseteq \mathbb{A}^1_{\mathbb{C}}$, such that the statement from (b) above is false.

<u>Example</u>



$$I(X_1)+I(X_2)=\langle x_2-x_1^2, x_2\rangle$$
$$=\langle x_1^2, x_2\rangle$$
$$I(X_1\cap X_2)=\sqrt{\langle x_1^2,x_2\rangle}$$
$$=\langle x_1,x_2\rangle$$

vanishing loci are equal

can remember that X_1, X_2 intersect with multiplicity 2

<u>Later</u> Generalize from affine varieties to affine schemes

$\rightsquigarrow$ Scheme-theoretic intersection of $X_1, X_2 \;\hat{=}\;$ ideal $\langle x_1^2, x_2\rangle$

Cor (Weak Nullstellensatz)

Let $J \subseteq K[x_1, \dots, x_n]$ be an ideal, then the following are equivalent:

(a) $V(J) = \emptyset$ (b) $J = K[x_1, \dots, x_n]$ (c) $1 \in J$.

In other words: given polynomials $f_1, f_2, \dots, f_r \in K[x_1 \dots x_n]$

either

$\exists$ solution a of $f_1(a) = f_2(a) = \dots = f_r(a)$

or

$\exists\, g_1, \dots, g_r \in K[x_1 \dots x_n]$: $g_1 f_1 + \dots + g_r f_r = 1$

this is the "Nullstelle" from the name of the theorem.

Functions on affine varieties

Big Picture

→ have some nice geometric spaces X given by affine varieties

→ Key insight of Grothendieck:
Now we need to understand functions on X!

Alexander Grothendieck

→ Other subjects in mathematics:

vector space V $\rightsquigarrow$ linear forms & dual space V^*

topological space S $\rightsquigarrow$ continuous functions $C(S)$

group G $\rightsquigarrow$ characters G^*

→ Now: affine variety $X \rightsquigarrow$??

<u>Def</u> Let $X \subseteq \mathbb{A}^n$ be an affine variety.
A <u>polynomial function</u> on X is a map $X \to K,\ x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$. Let

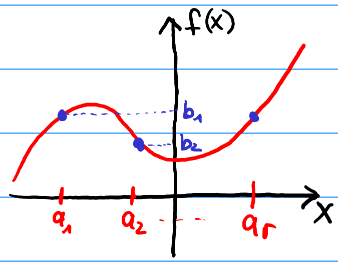
$$A(X) = \{ f : X \to K \ : \ f \text{ polynomial function} \}$$

be the <u>coordinate ring</u> of X.

<u>Lem</u> $A(X)$ is a K-algebra (with respect to pointwise $+$ and $\cdot$).

<u>Pf</u> f, g polyn. fct $\Rightarrow f+g, f \cdot g$ poly. fct.
$K \to A(X)$ sending $\lambda \in K$ to const. function $f(x) = \lambda$ is ring homom. $\square$

<u>Exa</u> $X = \{a_1, \dots, a_r\} \subseteq \mathbb{A}^1 \rightsquigarrow$ What is $A(X)$?
For any $b = (b_1, \dots, b_r) \in K^r$ there is a polynomial $f_b \in K[x]$ with $f_b(a_i) = b_i$ ($\rightsquigarrow$ Lagrange interpolation)
$\Rightarrow A(X) = K^r$ w/ componentwise addit./mult.



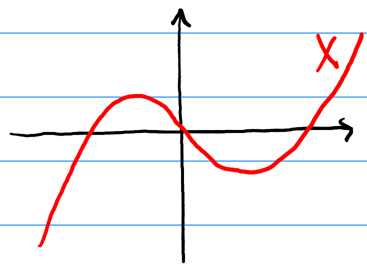
With the Nullstellensatz, we can describe $A(X)$ completely!

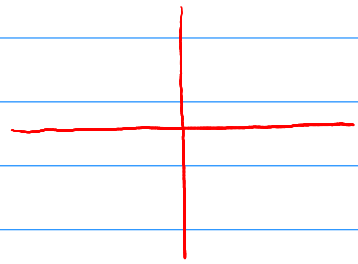
Prop Let $X = V(J) \subseteq \mathbb{A}^n$ be an affine variety.
Then the map
$$K[x_1,\dots,x_n] \longrightarrow A(X), \quad f \mapsto \left(\overset{X\to K}{x \mapsto f(x)}\right) \qquad (*)$$
is surjective with kernel $I(X) = \sqrt{J}$. In particular:
$$A(X) \cong K[x_1,\dots,x_n]/I(X). \qquad (**)$$

Proof
The map $(*)$ is surjective by definition of $A(X)$.
Assume f is in the kernel $\Leftrightarrow f(x) = 0 \quad \forall x \in X$
$\Leftrightarrow f \in I(X) \overset{\text{Nullstellensatz}}{=\!=} \sqrt{J}$.
Then $(**)$ follows from the isomorphism theorem. $\square$

Exa

$X = V(y - x^3 + x)$ $\qquad\qquad$ $X = V(xy)$





$A(X) = K[x,y]/\langle y - x^3 + x\rangle$
$\cong K[x]$

$A(X) = K[x,y]/\langle xy\rangle = \mathrm{span}_K\{1, x, x^2, \dots, y, y^2, \dots\}$
$= \{(f(x), g(y)) : f \in K[x], g \in K[y] : f(0) = g(0)\}$
functions on $V(y)$, $V(x)$ that agree at origin

Cor $A(\mathbb{A}^n) = K[x_1,\dots,x_n]$.
Pf $\mathbb{A}^n = V(\langle 0\rangle)$ so $I(\mathbb{A}^n) = \sqrt{\langle 0\rangle} = \langle 0\rangle$ and $A(\mathbb{A}^n) = K[x_1,\dots,x_n]/\langle 0\rangle$.
Slogan A polynomial in $x_1,\dots,x_n$ is determined uniquely by its values at points of $\mathbb{A}^n$.
Note Again it is crucial that we work over an alg. closed field!
Take $f(x) = x^2 + x \in \mathbb{F}_2[x] \rightsquigarrow \mathbb{A}^1_{\mathbb{F}_2} = \{\bar{0}, \bar{1}\}$ and $f(\bar{0}) = f(\bar{1}) = 0$
but $f \neq 0 \in \mathbb{F}_2[x]$!

Subvarieties and relative vanishing ideals

Above we used $K[x_1,\dots,x_n]$ to define affine varieties in $\mathbb{A}^n$ & vanishing ideals

Now we can use $A(Y)$ to generalize to subvarieties in Y & relative vanishing ideals

Construction Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety.

(a) For a subset $S \subseteq A(Y)$ we define its <u>zero locus</u> as

$$V(S) = V_Y(S) = \{x \in Y : f(x) = 0 \ \forall f \in S\}$$

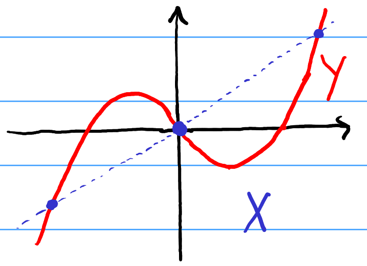
These subsets are called <u>affine subvarieties</u> of Y.

(b) For a subset $X \subset Y$, the <u>(vanishing) ideal of X in Y</u> is defined as

$$I(X) = I_Y(X) = \{f \in A(Y) : f(x) = 0 \ \forall x \in X\} \subseteq A(Y).$$

Exa

$Y = V(y - x^3 + x)$



$A(Y) = K[x,y]/\langle y - x^3 + x\rangle$

$S = \{[y - 3x]\} = \{[x^3 - 4x]\} = \{[x \cdot (x-2)(x+2)]\}$

$X = V_Y(S) = \{(-2,-6), (0,0), (2,6)\}$

Boring but instructive exercise

Let $Y \subseteq \mathbb{A}^n$ be an affine subvariety.

(a) Show that any subvariety $X \subseteq Y$ is also a subvariety of $\mathbb{A}^n$.

(b) Conversely, given a subvariety $X' \subseteq \mathbb{A}^n$ which satisfies $X' \subseteq Y$, prove that X' is a subvariety of Y.

(c) Prove that $I(X) \subseteq A(Y)$ is an ideal.

(d) Show that for $X \subseteq Y$ a subvariety, we have

$A(X) \cong A(Y) / I_Y(X)$.

makes sense since X is affine variety by (a).

(e)* Formulate the relative Nullstellensatz for $V_Y(-)$ and $I_Y(-)$.

In particular, what goes here:

$\{\text{affine subvarieties } X \subseteq Y\} \underset{?}{\overset{?}{\rightleftarrows}}$??

(f)** Prove the relative Nullstellensatz from the absolute version we saw above.

(g) Show the properties of $V_Y(-)$ and $I_Y(-)$ under $\cup$ and $\cap$ generalized from those of $V(-)$ and $I(-)$.

<u>Hints</u> (*) see [Gathmann, Remark 1.18]

(**) see [Gathmann, Exercise 1.23]

Products of affine varieties

We have already seen that for $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ affine varieties, we have that $X \times Y \subseteq \mathbb{A}^{n+m}$ is an affine variety. Let $I(X) \subseteq K[x_1, \dots, x_n]$ and $I(Y) \subseteq K[y_1, \dots, y_m]$ be their ideals, denote $R = K[x_1, \dots, x_n, y_1, \dots, y_m] = A(\mathbb{A}^{n+m})$ and define

$$I_{X \times Y} = I(X) \cdot R + I(Y) \cdot R \overset{\text{ideal}}{\subseteq} R.$$

Pro For $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ affine varieties, we have $I(X \times Y) = I_{X \times Y}$. The ring of polynomial functions on the product $X \times Y$ is given by

$$A(X \times Y) = A(X) \otimes_K A(Y).$$

Slogan Coordinate ring of product = tensor product of coordinate rings

General facts on tensor products

A, B rings with ring morphisms $C \rightrightarrows \begin{matrix} A \\ B \end{matrix}$

$\Rightarrow$ $A \otimes_C B$ ring with $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 \cdot b_2)$.

Exa $K[x_1, \dots, x_n, y_1, \dots, y_m] = K[x_1, \dots, x_n] \otimes_K K[y_1, \dots, y_m]$.

Lem $I_A \subseteq A$ and $I_B \subseteq B$ ideals

$\Rightarrow$ $I_{A,B} := I_A \cdot A \otimes_C B + I_B \cdot A \otimes_C B$

$:= \langle \{ i_a \otimes b : i_a \in I_A, b \in B \} \cup \{ a \otimes i_b : a \in A, i_b \in I_B \} \rangle \subseteq A \otimes_C B$

is an ideal and the map

$$(A/I_A) \otimes_C (B/I_B) \xrightarrow{\sim} A \otimes_C B / I_{A,B}, \quad (a + I_A) \otimes (b + I_B) \mapsto a \otimes b + I_{A,B}$$

is an isomorphism.

Pf Show map above is well-defined & write down inverse. $\square$

<u>Pf. of Pro</u> For $R = K[x_1, \dots, x_n, y_1, \dots, y_m]$, want:

$$I(X \times Y) = I(X) \cdot R + I(Y) \cdot R =: I_{X \times Y}.$$

Easy: $V(I_{X\times Y}) = X \times Y \subseteq \mathbb{A}^{n+m} \Rightarrow I(X \times Y) = \sqrt{I_{X\times Y}}$

Thus it suffices to show:

$I_{X\times Y}$ is radical $\Longleftrightarrow$ $R/I_{X\times Y}$ is <u>reduced</u>   0 is only nilpotent element. $(\star)$

$$R/I_{X\times Y} = K[x_1, \dots, x_n] \otimes_K K[y_1, \dots, y_m] / I(X) \cdot R + I(Y) \cdot R$$
$$\overset{\text{Lem}}{=} (K[x_1, \dots, x_n]/I(X)) \otimes_K (K[y_1, \dots, y_m]/I(Y)) \qquad (\star\star)$$
$$= A(X) \otimes_K A(Y).$$

<u>Fact</u> K perfect field and T, U reduced K-algebras
$\Rightarrow$ $T \otimes_K U$ reduced.

Since $I(X), I(Y)$ are radical $\Rightarrow A(X), A(Y)$ reduced
$\overset{\text{Fact}}{\Longrightarrow} A(X) \otimes_K A(Y)$ reduced, which proves $(\star)$.

Finally:

$A(X \times Y) = R/I(X \times Y) \overset{(\star\star)}{=} A(X) \otimes_K A(Y)$ already checked. $\square$